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Vacuum Cherenkov radiation and photon triple-splitting in a Lorentz-noninvariant extension of quantum electrodynamics

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Abstract

We consider a CPT–noninvariant scalar model and a modified version of quantum electrodynamics with an additional photonic Chern–Simons-like term in the action. In both cases, the Lorentz violation traces back to a spacelike background vector. The effects of the modified field equations and dispersion relations on the kinematics and dynamics of decay processes are discussed, first for the simple scalar model and then for modified quantum electrodynamics. The decay widths for electron Cherenkov radiation in modified quantum electrodynamics and for photon triple-splitting in the corresponding low-energy effective theory are obtained to lowest order in the electromagnetic coupling constant. A conjecture for the high-energy limit of the photon-triple-splitting decay width at tree level is also presented.

Key words: Lorentz violation, Quantum electrodynamics, Cherenkov radiation, Photon splitting
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1 Introduction

The Lorentz-noninvariant Maxwell–Chern–Simons (MCS) model in four spacetime dimensions has been extensively studied over the last years [1–3].

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With suitable interactions added, the model allows for photon triple-splitting ($\gamma \to \gamma\gamma\gamma$), but the decay width of the photon has only been calculated for one special case [4]. Another interesting process is vacuum Cherenkov radiation, with an electron emitting an MCS photon ($e^- \to e^-\gamma$), for which there exists an order of magnitude estimate of the decay probability [5].

In order to improve our understanding, we consider in the present article simple scalar models where Lorentz symmetry is broken by a (purely) spacelike background vector, and focus on the effects in decay processes. We, then, turn to two- and three-particle decays in a modified version of quantum electrodynamics (QED), which has an anisotropic photonic Chern–Simons-like term [1–3] added to the standard Maxwell–Dirac action of QED [6]. First, we calculate the vacuum Cherenkov radiation of a moving electron (i.e., moving with respect to the preferred frame from the Chern–Simons-like term). Second, we obtain the general result for photon triple-splitting in the low-energy effective theory with electrons integrated out (specifically, the photon model with the Euler–Heisenberg quartic interaction term added to the quadratic MCS terms). Third, we discuss the possible high-energy behavior of the photon-triple-splitting decay width at tree level, which may be compared to the behavior of the exact tree-level result for vacuum Cherenkov radiation.

The scope of the present article has been restricted to a relatively simple and well-understood model, as our goal is to perform a *complete* calculation of certain two- and three-particle decays, at least at tree level. In the pure photon sector, there are only two possible bi-linear Lorentz-violating terms [2], one of which, the CPT–odd Chern–Simons-like term, we study in detail. The other possible term, which is CPT–even, will be discussed briefly at the end of the article.

This article is organized as follows. In Section 2, we present the scalar and photon models considered and introduce the concept of effective mass square. In Section 3, we define the decay width in a Lorentz-violating theory and, in Section 4, study a few simple decay processes for scalars. In Section 5, we give results for Cherenkov radiation in modified QED with a spacelike Chern–Simons-like term and, in Section 6, for photon triple-splitting in the corresponding low-energy effective theory (details of the calculation are relegated to the appendices). In Section 7, we place our results in a larger context and speculate on photon triple-splitting from the CPT–even Lorentz-violating term mentioned above.

The present article is rather technical and the reader who is only interested in the main results may skip ahead to Sections 5 and 6, while referring back to Sections 2 and 3 for the necessary definitions.

2 Lorentz violation and effective mass squares

2.1 Framework and conventions

We work in four-dimensional Minkowski spacetime, and postulate translation invariance which implies energy-momentum conservation. The Lorentz-symmetry breaking is implemented by the presence of constant background tensors in the action, as in the Standard Model extension of Ref. [2]. We are mainly interested in bi-linear breaking terms which lead to modified dispersion relations and new definitions of the free particle states. As will be seen later, such modifications are more interesting than having just one particular type of Lorentz-violating interaction, since the changed kinematics appears in all kinds of physical processes.

In our examples, Lorentz invariance is broken by a real background four-vector

$$\zeta^\mu = (\zeta^0, \boldsymbol{\zeta}), \qquad \zeta^\mu \equiv m\,\widehat{\zeta}^{\,\mu}, \tag{2.1}$$

where we have split ζ^μ into a dimensionless unit vector $\widehat{\zeta}^{\,\mu} = (\widehat{\zeta}^{\,0}, \widehat{\boldsymbol{\zeta}})$ characterizing the Lorentz-breaking direction in spacetime (normalization $\widehat{\zeta}^{\,\mu}\widehat{\zeta}_\mu = \pm 1$) and a mass scale $m > 0$ setting its length. If ζ^μ is spacelike, there is a preferred spatial direction in coordinate frames with $\zeta^0 = 0$. For timelike ζ^μ, there is rotational invariance in frames with $\boldsymbol{\zeta} = 0$. We do not consider the "lightlike" case ($\zeta^\mu\zeta_\mu = 0$), which can be viewed as being of measure zero.

In the purely spacelike case ($\zeta^0 = 0$ and $|\widehat{\boldsymbol{\zeta}}| = 1$), we will use the following notations:

$$k_\parallel \equiv \mathbf{k}\cdot\widehat{\boldsymbol{\zeta}}, \quad \mathbf{k}_\perp \equiv \mathbf{k} - k_\parallel\,\widehat{\boldsymbol{\zeta}}, \quad k_\perp \equiv |\mathbf{k}_\perp|, \tag{2.2}$$

for an arbitrary three-vector $\mathbf{k}$. Throughout, we employ the Minkowski metric $(\eta_{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$, take $\epsilon_{0123} = 1$, and set $\hbar = c = 1$ (except when stated otherwise).

In the main part of this article, we assume that ζ^μ is spacelike and choose a particular coordinate frame so that ζ^μ is purely spacelike. The calculations are simplified in such a frame (loosely called the "purely spacelike frame"), because Lorentz invariance with respect to boosts in directions orthogonal to $\boldsymbol{\zeta}$ is preserved, together with rotational invariance around $\boldsymbol{\zeta}$. In Section 3.2, we discuss how to interpret these theories in a general frame.

2.2 Maxwell–Chern–Simons model

As mentioned in the Introduction, we start from a noninteracting photon model which results from adding a CPT–odd Abelian Chern–Simons-like term [1–3] to the standard Maxwell term [6]. The model action is then given by

$$\mathcal{S}_{\mathrm{MCS}} = \mathcal{S}_{\mathrm{M}} + \mathcal{S}_{\mathrm{CS}}\,, \tag{2.3}$$

with the following Maxwell and Chern–Simons-like terms:

$$\mathcal{S}_{\mathrm{M}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(-\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \right), \tag{2.4}$$

$$\mathcal{S}_{\mathrm{CS}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(\frac{1}{4}\; m\, \epsilon_{\mu\nu\rho\sigma}\, \widehat{\zeta}^{\,\mu}\, A^\nu\, F^{\rho\sigma} \right), \tag{2.5}$$

where $A_\mu(x)$ is the gauge field and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ the field strength. The action (2.5) is gauge invariant, provided the field strength vanishes rapidly enough at infinity.

The extra term (2.5) may, for example, be induced by the CPT anomaly of chiral gauge theories over a topologically nontrivial spacetime manifold [7–9]. From astrophysical bounds [1,2,10], the mass parameter m is known to be very small, at least for the regular photon and the present universe. The relatively simple model (2.3) may, however, have other applications and the intention of the present article is purely theoretical.

If the background vector ζ^μ is spacelike, quantization is possible in frames where this vector is purely spacelike [3]. In these frames, the invariance under time reversal T is broken, while charge conjugation C and parity reflection P remain symmetries; cf. Refs. [9,11,12]. For timelike ζ^μ, a quantization does not appear to be possible [3].

The field equations of the model (2.3) are

$$\Big(\Box\eta^{\mu\nu} - \partial^\mu\partial^\nu - \; m\, \epsilon^{\mu\nu\rho\sigma} \widehat{\zeta}_\rho\, \partial_\sigma \Big)\, A_\nu = 0. \tag{2.6}$$

From a plane-wave *Ansatz*, one obtains the dispersion relation for the momentum four-vector k^μ,

$$(k^\mu k_\mu)^2 + (k^\mu k_\mu)(\zeta^\nu \zeta_\nu) - (k^\mu \zeta_\mu)^2 = 0, \tag{2.7}$$

with $\zeta^\nu \equiv m\, \widehat{\zeta}^{\,\nu}$. This relation gives two different propagation modes, called $\oplus$ and $\ominus$ in the following, according to the form the dispersion relation takes in a purely spacelike frame,

$$\omega_\pm(\mathbf{k}) = \sqrt{\Big(\omega_{\parallel,\pm}(k_\parallel)\Big)^2 + k_\perp^2}\;, \tag{2.8}$$

with "parallel energies"

$$\omega_{\parallel,\pm}(k_\parallel) \equiv \sqrt{k_\parallel^2 + m^2/4} \,\pm\, m/2 \sim |k_\parallel| \pm m/2, \tag{2.9}$$

for $|k_\parallel| \gg m > 0$.

The $\oplus$ mode has timelike four-momentum and a mass gap. The $\ominus$ mode has spacelike four-momentum and is gapless. But, even though certain photon momenta can be spacelike, the theory is still causal [3]. In Appendix A, we present the modified polarization vectors of the photon, together with certain useful relations. For further details, see, in particular, Section 2 of Ref. [4].

2.3 Lorentz-noninvariant scalar model

The Lorentz-noninvariant photon model of the previous subsection is already quite complicated and, as a simpler example, we consider a complex scalar field with the following free action:

$$\mathcal{S}_{\text{scalar}} = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(\partial_\mu\bar{\phi}\,\partial^\mu\phi + \,\mathrm{i}\,m\,\bar{\phi}\,\widehat{\zeta}^\mu\partial_\mu\phi - \,\mathrm{i}\,m\,\phi\,\widehat{\zeta}^\mu\partial_\mu\bar{\phi} - M_\phi^2\,\bar{\phi}\phi \right), \tag{2.10}$$

with m and M_ϕ taken positive. A spacetime-dependent phase redefinition of the fields could eliminate the Lorentz-violating term in the free model; cf. Refs. [12,13]. Here, however, we assume that the interactions considered do not allow these phase redefinitions (see Section 4).

In this scalar model, the conventional discrete symmetries C, P, and T are broken [11]. The combinations PT, CP, CT are conserved, but not CPT.

The action (2.10) yields the field equation

$$\Box\phi - 2\,\mathrm{i}\,m\,\widehat{\zeta}^\mu\partial_\mu\phi + M_\phi^2\,\phi = 0, \tag{2.11}$$

and the complex-conjugate equation for $\bar{\phi}$. The dispersion relation for the momentum vector k^μ is then given by:

$$k^\mu k_\mu \pm 2m\,\widehat{\zeta}^\mu k_\mu - M_\phi^2 = 0, \tag{2.12}$$

with the upper sign for ϕ and the lower for $\bar{\phi}$.

Solving the quadratic k^0 equation (2.12), one has for the positive energy branch,

$$\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 \pm 2\mathbf{k}\cdot\boldsymbol{\zeta} + (\zeta^0)^2 + M_\phi^2} \mp \zeta^0, \tag{2.13}$$

with $\zeta^\mu \equiv m\,\widehat{\zeta}^\mu$. For the purely spacelike case, the energy reduces to

$$\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 \pm 2\mathbf{k}\cdot\boldsymbol{\zeta} + M_\phi^2} = \sqrt{\omega_\parallel(k_\parallel)^2 + k_\perp^2}, \tag{2.14}$$

with

$$\omega_\parallel(k_\parallel) \equiv \sqrt{k_\parallel^2 \pm 2mk_\parallel + M_\phi^2} \sim |k_\parallel| \pm m \;\mathrm{sgn}\, k_\parallel, \tag{2.15}$$

for $|k_\parallel| \gg \max(m, M_\phi)$. The asymptotic parallel energy (2.15) of the scalar model has the same structure as the result (2.9) of the MCS model, apart from the sign of the mass term (remember that C, P, and T are broken in the scalar model, whereas only T is broken in the purely spacelike MCS model).

For $M_\phi^2 + \zeta^\mu\zeta_\mu \geq 0$, the group velocity $|\partial\omega/\partial\mathbf{k}|$ is less or equal to 1; cf. Ref. [13]. Microcausality (locality) can then be verified by a calculation analogous to the Lorentz-invariant case. Recall that locality [6] requires, in particular, a vanishing commutator $[\phi(x), \bar{\phi}(y)]$ for spacelike separation $x^\mu - y^\mu$.

The Lorentz-violating terms in (2.3) and (2.10) have a similar structure: each term is bi-linear in the fields, has a single derivative, and couples to the background vector. The models also have similar dispersion relations $\omega(\mathbf{k})$, as long as the scalar mass parameter M_ϕ does not dominate, $M_\phi \ll |\mathbf{k}|$. A photon mass term in the MCS model is, of course, forbidden by gauge invariance.

2.4 Effective mass squares

One way to look at modified dispersion relations is to think of the norm square of the energy-momentum four-vector as the "effective mass square" of the particle with corresponding three-momentum,

$$M_{\text{eff}}^2(\mathbf{k}) \equiv \omega(\mathbf{k})^2 - |\mathbf{k}|^2 = k^\mu k_\mu. \tag{2.16}$$

This effective mass square is associated with a specific mode and may depend on the coordinate frame used. [1] For timelike momentum, on the one hand, the effective mass square is the energy square in the frame with zero three-momentum. For spacelike momentum, on the other hand, the effective mass square is negative and there is no frame with vanishing three-momentum.

The effective mass square of the photon from the purely spacelike MCS model (2.3) is given by

$$M_{\text{eff, MCS}\,\pm}^2(\mathbf{k}) = \pm\, m\, \omega_{\parallel,\pm}(k_\parallel) \;= \pm m\, |k_\parallel| + m^2/2 + \mathrm{O}\Big(m^3/|k_\parallel|\Big), \tag{2.17}$$

for $|k_\parallel| \gg m$. The absolute value of this effective mass square grows with the momentum component in the preferred direction, but is still suppressed

[1] The concept of effective mass square appears already in Refs. [14,15] and, more recently, has been used in Ref. [16].

relative to the energy square,

$$\frac{|M^2_{\text{eff, MCS}\pm}|}{\omega^2_\pm} \leq \frac{|M^2_{\text{eff, MCS}\pm}|}{\omega^2_{\|,\pm}} \sim \frac{m}{|k_\||}\,, \tag{2.18}$$

for $|k_\|| \gg m$.

The effective mass square of the scalar model (2.10) in the purely spacelike frame is given by

$$M^2_{\text{eff, scalar}}(\mathbf{k}) = \pm 2\, m\, k_\| + M^2_\phi, \tag{2.19}$$

with the upper sign for ϕ and the lower for $\bar{\phi}$. The (anti-)scalar effective mass square from Eq. (2.19) has the same structure as the asymptotic result (2.17) of the MCS model, apart from the sign of the linear term.

As will be seen later, these effective mass squares appear directly in the transition amplitudes or, at least, determine their strength. But they even occur in *free* theories and might, for example, play a role in Lorentz-noninvariant neutrino oscillations due to Fermi-point splitting [17]. In this case, the dispersion relation of the neutrino is essentially the same as the one of our scalar model, Eq. (2.13). As the effective mass squares (and their differences) increase linearly with energy, the usual energy-dependence of the oscillations is canceled. The momentum dependence of the effective mass squares might, therefore, play an important role in neutrino oscillations, provided the usual Lorentz-invariant masses are small enough.

3 Decay width and coordinate-frame independence

3.1 Decay width and decay parameter

For the decay of a particle with momentum $\mathbf{q}$ and dispersion relation $\omega(\mathbf{q})$ into n particles with momenta $\mathbf{k}_i$ and dispersion relations $\omega_i(\mathbf{k}_i)$, the width can be defined as follows [18]:

$$\Gamma(\mathbf{q}) \equiv \frac{1}{\sigma}\frac{1}{N(\mathbf{q})}\int\left(\prod_{i=1}^{n}\frac{\mathrm{d}^3k_i}{(2\pi)^3\,N_i(\mathbf{k}_i)}\right) \times (2\pi)^4\delta^3\Big(\mathbf{q}-\sum_j \mathbf{k}_j\Big)\,\delta\Big(\omega(\mathbf{q})-\sum_l \omega_l(\mathbf{k}_l)\Big)\,|A(\mathbf{q},\omega,\mathbf{k}_i,\omega_i)|^2, \tag{3.1}$$

with symmetry factor σ for identical decay products and transition amplitude A. In this expression, N and N_i are normalization factors which, in the Lorentz-invariant case, are given by 2ω and $2\omega_i$ with corresponding normalization of the amplitude.

We find the following normalization factor for the Lorentz-violating scalar model (2.10):

$$N = 2\left(\omega \pm m\,\widehat{\zeta}^{0}\right) \geq 0, \tag{3.2}$$

where the (upper) lower sign is for the field (ϕ) $\bar{\phi}$. For the MCS model (2.3) in the purely spacelike frame, the normalization factor reduces to the usual one,

$$N = 2\omega, \tag{3.3}$$

but with ω given by expression (2.8).

The "decay parameter" γ is now defined by

$$\gamma \equiv N(\mathbf{q})\,\Gamma(\mathbf{q}). \tag{3.4}$$

In a Lorentz-invariant theory, γ does not depend on the initial three-momentum $\mathbf{q}$ but only on the masses of the particles involved. The reason is that the only Lorentz invariants available are functions of the masses.[2] In a Lorentz-violating theory, the decay parameter may also depend on the momentum $\mathbf{q}$ through contractions of q^μ with the background tensors.

In most cases, it suffices to calculate γ in a specific frame and to generalize to the coordinate-independent expression. The calculation will be especially simple for two types of coordinate frames:

(1) the class of frames with *vanishing* three-momentum of the decaying particle, $\mathbf{q} = \mathbf{0}$, for the case of timelike four-momentum q^μ and arbitrary background vector ζ^μ;

(2) the class of frames where the background vector is *purely* spacelike or timelike, for the case of a spacelike or timelike background vector ζ^μ.

For example, if the calculation in our models is performed in a frame where ζ^μ is purely spacelike, γ will be a function of $\mathbf{q}\cdot\boldsymbol{\zeta}$, which has the frame-independent form $-q^\mu\zeta_\mu$.

3.2 Lifetime and quasi-restmass

It appears reasonable to demand that, at least in first approximation, the "lifetime" of an unstable particle, defined by

$$T(\mathbf{q}) \equiv 1/\Gamma(\mathbf{q}), \tag{3.5}$$

[2] The use of the decay parameter (or, rather, the decay constant) instead of the decay width in the rest frame is only necessary if one has massless particles; cf. Refs. [19,20].

transforms as a genuine time with respect to changes of frame (i.e., as the zero-component of a four-vector). Then, the quantity T must change as follows:

$$T(\mathbf{q}',\boldsymbol{\beta}) = \frac{1}{\sqrt{1-|\boldsymbol{\beta}|^2}}\Big(1-\boldsymbol{\beta}\cdot\mathbf{v}(\mathbf{q})\Big)\,T(\mathbf{q}), \tag{3.6}$$

with $\mathbf{v}$ the velocity of the particle [tentatively identified with the group velocity $\mathbf{v}_g \equiv \partial\omega/\partial\mathbf{q}$ for energy $\omega(\mathbf{q})$ in the old frame], $\mathbf{q}$ and $\mathbf{q}'$ the momenta in the old and new frame, and $\boldsymbol{\beta}$ the boost velocity characterizing the motion of the new frame relative to the old. Equation (3.6) can be checked explicitly in the models of Section 2, provided the group velocity is used for $\mathbf{v}$.

As long as $|\mathbf{v}_g(\mathbf{q})| < 1$, there always exists a "rest frame" with $\mathbf{v}'_g = \mathbf{0}$ and

$$T\big|_{\text{rest}} \equiv T(\mathbf{q},\mathbf{v}_g(\mathbf{q})) = \sqrt{1-|\mathbf{v}_g(\mathbf{q})|^2}\;T(\mathbf{q}) \leq T(\mathbf{q}). \tag{3.7}$$

One can take $T|_{\text{rest}}$, the minimum of the lifetimes over all frames, as the definition of the observer-independent lifetime of that specific mode. This lifetime will then be properly time-dilated in frames where the decaying particle is moving.

For Lorentz-invariant theories, the rest-frame decay width can be written as

$$\Gamma_{\text{rest}} \equiv \left(T\big|_{\text{rest}}\right)^{-1} = \frac{1}{2M_{\text{rest}}}\,\gamma, \tag{3.8}$$

where M_{rest} is simply the mass of the decaying particle. By analogy, we define the mode-dependent "quasi-restmass"

$$M_{\text{rest}} \equiv \frac{1}{2}N(\mathbf{q})\,\sqrt{1-|\mathbf{v}_g(\mathbf{q})|^2}, \tag{3.9}$$

so that (3.8) also holds for Lorentz-violating theories. Note that there is no direct relation between this mass and the effective mass square as discussed in Section 2.

For the Maxwell–Chern–Simons model (2.3), we obtain

$$M_{\text{rest}}^{(\text{MCS})} = \frac{m}{2}\left(1 \pm \frac{m^2}{\sqrt{m^4+4\,(q^\mu\zeta_\mu)^2}}\right), \tag{3.10}$$

which is always less or equal than m. With $|q^\mu\zeta_\mu|$ increasing from zero to infinity, $M_{\text{rest}}^{(\text{MCS})}$ interpolates monotonically between 0 and $m/2$ for the $\ominus$ mode and between m and $m/2$ for the $\oplus$ mode. For the scalar model (2.10),

the quasi-restmass is constant,

$$M_{\text{rest}}^{(\text{scalar})} = \sqrt{M_\phi^2 + \zeta^2}. \tag{3.11}$$

In the remainder of this article, we will focus on calculating the decay parameter γ as defined by Eq. (3.4), which, as explained above, may become momentum-dependent in Lorentz-noninvariant theories.

4 Lorentz-noninvariant decay processes of scalars

4.1 Preliminaries

In this section, we show how the effective mass squares of Section 2.4 enter in a simple setting. The model used is the scalar model defined by the action (2.10).

We, first, consider decay processes with two particles in the final state. In a Lorentz-invariant theory, the phase-space integral would then be trivial because the transition amplitude square is constant. Also, we start with scalar particles in order to avoid the complications of fields with spin. [For particles with spin, different modes are split and polarization identities, such as Eq. (A.6) in the first appendix, may introduce the Lorentz-breaking tensors explicitly into the amplitude; see, e.g., Eq. (5.8) below.]

In addition to the Lorentz-violating particle ϕ with mass M_ϕ, we introduce a real Lorentz-invariant scalar particle ψ with mass M_ψ. Its free action is given by

$$\mathcal{S}_{\psi^2} = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(\frac{1}{2}\, \partial_\mu \psi\, \partial^\mu \psi - \frac{1}{2}\, M_\psi^2\, \psi^2 \right). \tag{4.1}$$

Purely for illustrative purposes, we will employ *ad hoc* Lorentz-invariant interactions between these two particles, which can, for example, be obtained by integrating out another heavy Lorentz-invariant particle. The corresponding coupling constants (generically denoted g) are assumed to be sufficiently small, so that the tree-level results can be used in a first approximation.

Regarding Lorentz-noninvariance effects in tree-level decays, there are, in general, three cases to consider:

(1) only the decay particle violates Lorentz invariance;
(2) one or more of the decay products violate Lorentz invariance, but not the decay particle;
(3) both the decay particle and at least one of the decay products violate Lorentz invariance.

We will discuss these three cases in turn.

There is also the possibility of having explicit Lorentz violation in the interaction Hamiltonian, but this is of minor interest here, as we intend to study how modified dispersion relations control physical interaction processes which could naively be expected to be Lorentz invariant.

4.2 Two-particle decay of ϕ into ψ's

In a general Lorentz-invariant scalar theory, the two-particle decay $\phi \to \psi\psi$ is kinematically allowed if the mass of the decaying particle is larger than the sum of the masses of the produced particles. But, for the Lorentz-violating model considered, it is really the effective mass which determines the threshold. With $M_{\text{eff}}(q_\parallel) = \sqrt{2mq_\parallel + M_\phi^2} > 2M_\psi$, the following inequality must hold in the purely spacelike frame:

$$mq_\parallel > 2M_\psi^2 - M_\phi^2/2\,. \tag{4.2}$$

Because only the decaying particle has Lorentz violation, the squared transition amplitude $|A|^2$ remains constant over the two-particle phase-space, as in the usual case. Hence, the decay parameter is simply the product of $|A|^2$ and the two-particle phase-space volume,

$$V = \frac{\lambda(M_{\text{eff}}^2, M_\psi^2, M_\psi^2)}{2!\; 8\pi M_{\text{eff}}^2}, \tag{4.3}$$

with the Källen function $\lambda(x, y, z) \equiv \left(x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\right)^{1/2}$; cf. Ref. [18]. The phase-space volume V depends only weakly on M_{eff}^2 if the mass M_ψ of the decay products is small. All in all, the decay width in the purely spacelike frame is given by

$$\Gamma = \frac{1}{2\omega}\,\gamma, \quad \gamma = V\,|A|^2. \tag{4.4}$$

One observes that the decay width behaves as if the decaying particle had a Lorentz-invariant mass square equal to the effective mass square M_{eff}^2 as defined by Eq. (2.19). [The decay width has a factor $1/(2\omega + 2\zeta^0)$ instead of $1/(2\omega)$ for an arbitrary spacelike frame.] But, using Eqs. (3.8) and (3.11), the decay width *at rest* is found to be given by

$$\Gamma_{\text{rest}} = \frac{1}{2\sqrt{M_\phi^2 - m^2}}\, V\,|A|^2, \tag{4.5}$$

which is larger by a factor $M_{\text{eff}}/\sqrt{M_\phi^2 - m^2}$ than the width of a Lorentz-

invariant particle with mass square equal to M_{eff}^2. The reason is simply that the particle ϕ moves with a velocity that is different from the one of a Lorentz-invariant particle with the same momentum $\mathbf{q}$ and energy ω for a mass-square value equal to $M_{\text{eff}}^2(\mathbf{q})$.

Next, consider special interactions of the type

$$\mathcal{S}_{\phi\psi^2} = \frac{g}{2!}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \Big((\Box^l\phi)\psi^2 + \text{H.c.}\Big)\,, \tag{4.6}$$

for an integer $l \geq 1$ and a real coupling constant g of mass dimension $1-2l$. Provided condition (4.2) holds, we then have

$$\gamma(\mathbf{q}) = g^2\, V\, M_{\text{eff}}^{4l}(\mathbf{q}). \tag{4.7}$$

For these special interactions, the decay width depends strongly on M_{eff}^2 and, thus, on the momentum component in direction $\widehat{\boldsymbol{\zeta}}$. This dependence is, however, accompanied by the same power of the small Lorentz-violating parameter m, according to Eq. (2.19).

The decay parameter (4.7), given in terms of the effective mass square (2.19), implies a different lifetime of the scalar and antiscalar particle (at least, at tree level), which is consistent with CPT noninvariance [11]. Lorentz noninvariance is manifest from the nontrivial momentum dependence of $\gamma(\mathbf{q})$.

4.3 Two-particle decay of ψ into ϕ's

We, now, introduce another special interaction term,

$$\mathcal{S}_{\psi\phi^2} = \frac{g}{2!}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \Big(\psi(\Box^l\phi)\phi + \text{H.c.}\Big)\,, \tag{4.8}$$

between the standard scalar particle ψ and two Lorentz-violating particles ϕ. Interaction terms with d'Alembertian operator $\Box$ on ψ would just produce factors M_ψ^2 by use of the field equation for ψ. Also, we do not consider other types of second derivatives on the two ϕ fields.

The decay $\psi \to \phi\phi$ is only possible if the three-momentum $\mathbf{q}$ of ψ satisfies

$$q_\parallel \leq \frac{M_\psi^2 - 4M_\phi^2}{4m}. \tag{4.9}$$

The phase-space integration is difficult in a general frame. We can, however, evaluate the phase-space integral in a purely spacelike frame where the integral reduces to the standard one. In this frame, the invariance with respect to boosts in a direction orthogonal to $\boldsymbol{\zeta}$ allows us to focus on the case of decay momentum $\mathbf{q} \parallel \boldsymbol{\zeta}$, because γ cannot depend on $\mathbf{q}_\perp$. Then, we can reduce

the phase-space integral to a one-dimensional integral over the momentum component $k_\parallel$ of one of the decay products,

$$\gamma(\mathbf{q}) = \frac{1}{2!\,8\pi\,\sqrt{q_\parallel^2 + M_\psi^2}}\;\int \mathrm{d}k_\parallel\;|A|^2\Big|_{k_\perp = k_{\perp,0}}\,, \tag{4.10}$$

where the integration domain is determined by energy conservation and $k_{\perp,0}$ is a function of the parallel components, whose explicit form is not needed if the tensor structure of the integrand is known.

For the scalar model considered, the amplitude is still a function of the effective masses. But these effective masses are not constant over the available phase-space. The resulting decay parameter will, in general, not be a constant, even though the decaying particle has a Lorentz-invariant dispersion relation.

For the interaction (4.8), we find, in fact, the following decay parameter:

$$\gamma(\mathbf{q}) \sim g^2\, m^{2l}\, q_\parallel^{2l}, \tag{4.11}$$

which is consistent with the naive assumption of counting the number of derivatives in the interaction.

4.4 Splitting of one ϕ into two or more ϕ's

The ϕ particle is also unstable against self-splitting, provided

$$q_\parallel \leq -\frac{3M_\phi^2}{2m}\,. \tag{4.12}$$

In this case, the Lorentz noninvariance of the decay parameter has two origins: first, the decaying particle at the initial three-momentum $\mathbf{q}$ and, second, the decay products averaged over the allowed phase space.

With the special (charge-nonconserving) interaction

$$\mathcal{S}_{\phi\bar\phi^2+\bar\phi\phi^2} = \frac{g}{2!}\int_{\mathbb{R}^4} \mathrm{d}^4x\,\Big(\bar\phi^2\,\Box^l\phi - \phi\,\bar\phi\,\Box^l\bar\phi \;+\; \mathrm{H.c.}\Big)\,, \tag{4.13}$$

the transition amplitude square for double-splitting becomes

$$|A|^2 \sim g^2\,\Big(M_{\text{eff}}^2(\mathbf{q})^l + (-M_{\text{eff}}^2(\mathbf{k}))^l + (-M_{\text{eff}}^2(\mathbf{q}-\mathbf{k}))^l\Big)^2\,. \tag{4.14}$$

For $l \geq 2$, one obtains after integration $\gamma \sim g^2\, m^{2l}\, q_\parallel^{2l}$, just as expected from counting the powers of $q_\parallel$ and $k_\parallel$ in (4.14). For $l = 1$, the leading term in the amplitude cancels out and γ is a constant.

4.5 Summary

In this section, we have seen that effective mass squares, introduced in Section 2.4, determine how the fundamental Lorentz-symmetry breaking feeds into physical interaction processes. In many cases, one can also see, from the derivative structure of the interaction Lagrangian, which powers of the Lorentz-violating parameters appear in the transition amplitude and decay width.

5 Vacuum Cherenkov radiation in modified QED

After the toy models of the previous section, we turn to more interesting decay processes involving Maxwell–Chern–Simons (MCS) photons. There are, of course, no two-particle decays in the free MCS theory and we must add charged particles (at least, if we wish to keep the theory renormalizable).

We, therefore, enlarge the Maxwell–Chern–Simons model by adding conventional electrons. The relevant action is then given by

$$\mathcal{S}_{\text{QED+spacelike CS–term}} = \mathcal{S}_{\text{QED}} + \mathcal{S}_{\text{CS},\,\widehat{\zeta}^{\mu}\widehat{\zeta}_{\mu}=-1}\,, \tag{5.1}$$

with the Chern–Simons-like term (2.5) for spacelike $\widehat{\zeta}^{\mu}$ added to the standard quantum electrodynamics (QED) action [6],

$$\mathcal{S}_{\text{QED}} = \mathcal{S}_{\text{M}} + \mathcal{S}_{\text{D}}\,, \tag{5.2}$$

consisting of the Maxwell term (2.4) and the Dirac term

$$\mathcal{S}_{\text{D}} = \int_{\mathbb{R}^4} \mathrm{d}^4x\; \bar{\psi}\,(\mathrm{i}\,\gamma^{\mu}\partial_{\mu} - M - e\,\gamma^{\mu}A_{\mu})\,\psi, \tag{5.3}$$

where the electron from field $\psi(x)$ has charge e and mass $M > m/2 > 0$. As before, the two polarization modes of the MCS photon are denoted $\oplus/\ominus$, corresponding to the $+/-$ sign in the dispersion relation (2.8).

The kinematics and interactions allow for pair creation, $\oplus \to e^+e^-$, which has been studied previously in Ref. [14]. But, this process has a threshold: pair creation is only possible for large enough photon momentum in the purely spacelike preferred frame,

$$|q_{\parallel}| \geq \frac{2M\sqrt{4M^2 - m^2}}{m} \sim \frac{4M^2}{m}, \tag{5.4}$$

which corresponds to having an effective photon mass larger than twice the electron mass. This photon momentum is, however, far beyond the Planck

scale,

$$|q_\parallel| \gtrsim 10^{45}\ \text{eV} \left(\frac{M}{511\ \text{k eV}}\right)^2 \left(\frac{10^{-33}\ \text{eV}}{m}\right), \tag{5.5}$$

for electron mass M and a "realistic" value for Chern–Simons scale m [1,2].

Photon decay into neutrinos would be an interesting alternative, if at least one neutrino mass state is strictly massless or has a mass M of at most 10^{-7} eV (corresponding, for $m = 10^{-33}$ eV, to a threshold of some 4×10^{19} eV, close to the highest known energy of cosmic rays). But this decay amplitude only arises at one loop, as the neutrino has no electric charge.

The Cherenkov process $e^- \to \ominus\, e^-$, on the other hand, occurs already at tree level and is allowed for *any* three-momentum $\mathbf{q}$ of the electron, provided $\mathbf{q}\cdot\widehat{\boldsymbol{\zeta}} \neq 0$. (See, e.g., Refs. [21,22] for a general discussion of vacuum Cherenkov radiation and Ref. [5] for a discussion in the context of the MCS model.) The tree-level amplitude A for this process follows directly from the QED interaction,

$$A = \bar{u}(q-k)\, \bar{\epsilon}_\mu(k)\, (-e\gamma^\mu)\, u(q), \tag{5.6}$$

with u the incoming and $\bar{u}$ the outgoing spinor and $\bar{\epsilon}_\mu$ the conjugate polarization vector of the MCS photon. The corresponding Feynman diagram is shown in Fig. 1 (the Feynman rules of standard QED are given in, e.g., Refs. [6,18]). Remark that the process $e^- \to \oplus\, e^-$ is not allowed kinematically. Note, furthermore, that we restrict ourselves to tree-level calculations in the theory (5.1) with Lorentz violation in the photon sector, but loops involving MCS photons will probably induce Lorentz violation also in the fermion sector.

After some γ-matrix algebra and averaging (summing) over initial (final) spinor polarizations, one obtains

$$\frac{1}{2}\sum_{\text{spins}} |A|^2 = e^2\, \left(4q^\mu q^\nu - 2q^\mu k^\nu - 2q^\nu k^\mu + 2q^\beta k_\beta\, \eta^{\mu\nu}\right)\, \bar{\epsilon}_\mu \epsilon_\nu. \tag{5.7}$$

Inserting the polarization identity (A.6) for $\bar{\epsilon}_\mu \epsilon_\nu$ and using the on-shell relation $2q^\beta k_\beta = k^2$ together with the photon dispersion law (2.7), this sum simplifies to

$$\frac{1}{2}\sum_{\text{spins}} |A|^2 = \frac{e^2}{2k^2+\zeta^2}\left(k^2(\zeta^2 - 4M^2) - 2(k\cdot\zeta)^2 + 4(q\cdot\zeta)(k\cdot\zeta) - 4(q\cdot\zeta)^2\right). \tag{5.8}$$

Here, a new feature is seen to enter two-particle decay: the Lorentz-violation parameter ζ^μ is introduced explicitly into the amplitude by the modified photon polarization identity. Equation (5.8) contains moreover the contraction

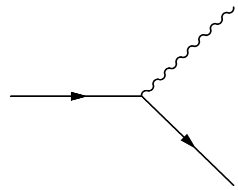

Fig. 1. Feynman diagram contributing to vacuum Cherenkov radiation (see text).

$q^\mu \zeta_\mu$, although the electron itself has no direct Lorentz violation. Hence, it is not true that the transition amplitude square for a two-particle decay process always reduces to a function of the effective mass squares.

In the coordinate frame with $\zeta^0 = 0$ and for initial electron momentum $\mathbf{q}$ parallel to $\boldsymbol{\zeta}$, the phase-space integral can be reduced in the same way as for the scalar models of the previous section. One finds

$$\gamma(q_\parallel) = \frac{1}{8\pi\,\sqrt{q_\parallel^2 + M^2}} \int_0^{k_{\max}} \mathrm{d}k_\parallel\ \frac{1}{2}\sum_{\text{spins}} |A|^2 \Big|_{k_\perp = k_{\perp,0}}, \tag{5.9}$$

where $k_{\perp,0}$ is a known function of $q_\parallel$ and $k_\parallel$ and the integration of the photon momentum component $k_\parallel$ runs from zero to a (positive or negative) value $k_{\max}$ defined by

$$k_{\max}(q_\parallel) \equiv \frac{2mq_\parallel\left(m + 2\sqrt{q_\parallel^2 + M^2}\right)}{m^2 + 4M^2 + 4m\sqrt{q_\parallel^2 + M^2}}\,. \tag{5.10}$$

In the coordinate frame with $\zeta^0 = 0$ and $|\boldsymbol{\zeta}| = m$, the decay width is then

$$\Gamma(\mathbf{q}) = \frac{1}{2\sqrt{|\mathbf{q}|^2 + M^2}}\ \gamma(q_\parallel). \tag{5.11}$$

Equation (5.9) can be integrated analytically and the decay parameter becomes

$$\begin{aligned}\gamma(q_\parallel) =& \frac{\alpha}{16\sqrt{q_\parallel^2 + M^2}} \Bigg[2m|k_{\max}|\sqrt{m^2 + 4k_{\max}^2} - 8m|q_\parallel|\left(\sqrt{m^2 + 4k_{\max}^2} - m\right) \\ & - 4\left(m^2 + 4M^2\right)|k_{\max}| + m\left(m^2 + 8M^2 + 16q_\parallel^2\right)\operatorname{arcsinh}\left(\frac{2|k_{\max}|}{m}\right)\Bigg],\end{aligned} \tag{5.12}$$

with fine-structure constant $\alpha \equiv e^2/(4\pi)$ and $k_{\max}$ from Eq. (5.10).

For $0 \leq |q_\parallel| < M$, the result (5.12) can be expanded in m/M,

$$\gamma(q_\parallel) = (4/3)\,\alpha\, m\,|q_\parallel|^3/M^2 + \mathrm{O}\left(\alpha\, m^2\,|q_\parallel|^3/|M|^3\right), \tag{5.13}$$

while, for $|q_\parallel| \gg M$, an expansion in $m/|q_\parallel|$ and $M/|q_\parallel|$ gives

$$\gamma(q_\parallel) = \alpha\, m\,|q_\parallel|\left(\ln(|q_\parallel|/m) + 2\ln 2 - 3/4\right) + \cdots, \tag{5.14}$$

where the ellipsis stands for subdominant terms. Hence, the decay parameter of the electron grows approximately linearly with the momentum component in the preferred direction, but is suppressed by one power of m, unlike the result in other theories where the decay rate increases more strongly above threshold [21,22].

For a relativistic electron, $|\mathbf{q}| \gg M$, the Cherenkov amplitude (5.6) preserves the helicity of the electron. While the complete decay parameter γ turns out to be different for left- and right-handed electrons, the leading term proportional to $\alpha\, m\,|q_\parallel|\,\ln(|q_\parallel|/m)$ is equal for both helicities.

The emitted $\ominus$ photon is approximately left- or right-circularly polarized (depending on the sign of $k_\parallel$) for momentum component $|k_\parallel| \gg m$, while, for $|k_\parallel|/m \to 0$, its polarization becomes linear [9]. Combined with electron helicity conservation, this implies that the Cherenkov decay violates angular momentum conservation by approximately one unit for large photon momentum component $|k_\parallel|$. This is reflected in the $|q_\parallel| \gg M$ transition amplitude which is strongly peaked at $|k_\parallel| \lesssim m$.

The vacuum Cherenkov decay process has also been studied quantum mechanically in Ref. [5] for modified QED (5.1) with a "lightlike" Chern–Simons-like term. [3] An estimate for the decay width of an electron at rest (ζ^μ is lightlike) has been obtained [5] by effectively expanding in $|\,\zeta^0/M| = |\,q^\mu\zeta_\mu|/q^2$,

$$\Gamma_{\text{rest}}^{\text{(QED+lightlike CS–term)}} \overset{?}{\sim} \kappa\,\frac{\alpha}{2M}\,|\,q^\mu\zeta_\mu|, \tag{5.15}$$

with an unknown constant κ. This behavior agrees with the expansion in $|\,q^\mu\zeta_\mu|/q^2$ of the exact result (5.12). In the rest frame of the electron, our result gives, namely,

$$\Gamma_{\text{rest}}^{\text{(QED+spacelike CS–term)}} \sim \frac{\alpha}{2M}\,|\,q^\mu\zeta_\mu|\left(\ln(|\,q^\mu\zeta_\mu|/m^2) + 2\ln 2 - 3/4\right). \tag{5.16}$$

[3] For the purely spacelike case, the authors of Ref. [5] have furthermore calculated the classical radiation rate, effectively in the limit $M \to \infty$. They obtain a rate proportional to $Q^2\,|\boldsymbol{\zeta}_{\text{class}}|^2$, where Q is the classical charge and $\boldsymbol{\zeta}_{\text{class}}$ the classical Chern–Simons parameter with dimension of inverse length. Our calculation reproduces their result (21) with corrections of order $Q^2\,|\boldsymbol{\zeta}_{\text{class}}|^2\, m/M$, for $m = \hbar\,|\boldsymbol{\zeta}_{\text{class}}|/c$.

Equation (5.16) contains an additional logarithmic dependence on $|\, q^{\mu}\zeta_{\mu}|$, which may, however, be absent for the lightlike MCS model because a lightlike vector ζ_{μ} does not define a mass scale m.

6 Photon triple-splitting in modified QED

At last, we turn to photon triple-splitting from the purely spacelike MCS model (2.3), continuing the work of Ref. [4]. There are eight decay channels, corresponding to all possible combinations of the different modes $\oplus$ and $\ominus$ from dispersion relation (2.8). In Appendix B, we show that the following three cases are allowed for generic initial three-momentum $\mathbf{q}$: $\oplus \to \ominus\ominus\ominus$, $\oplus \to \oplus\ominus\ominus$, and $\ominus \to \ominus\ominus\ominus$, whereas the five others are kinematically forbidden. For special momentum $\mathbf{q} \perp \boldsymbol{\zeta}$, the decay rate turns out to be zero except for the case of $\oplus \to \ominus\ominus\ominus$.

The implication would be that, with suitable interactions, all MCS photons are generally unstable against splitting. The exception would be for the lower-dimensional subset of $\ominus$ modes with three-momenta orthogonal to $\boldsymbol{\zeta}$.

Following Ref. [4], the interaction is taken to be the Euler–Heisenberg interaction and the photonic action considered reads

$$\mathcal{S}_{\text{photon}} = \mathcal{S}_{\text{MCS},\, \widehat{\zeta}^{\mu}\widehat{\zeta}_{\mu}=-1\,,\, \widehat{\zeta}^{0}=0} + \mathcal{S}_{\text{EH}}, \tag{6.1}$$

consisting of the MCS quadratic term (2.3), for purely spacelike background four-vector $\widehat{\zeta}^{\mu}$, and the quartic Euler–Heisenberg term

$$\mathcal{S}_{\text{EH}} = K \int_{\mathbb{R}^4} \mathrm{d}^4x \, \left[\left(\frac{1}{2}\, F_{\mu\nu}F^{\mu\nu} \right)^2 + 7 \left(\frac{1}{8}\, \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \right)^2 \right], \tag{6.2}$$

with effective coupling constant

$$K \equiv \frac{2\alpha^2}{45M^4}, \tag{6.3}$$

given in terms of the fine-structure constant $\alpha \equiv e^2/(4\pi) \approx 1/137$ and the electron mass M. In the modified version of quantum electrodynamics (QED) with action (5.1), the Euler–Heisenberg term arises from the low-energy limit of the one-loop electron contribution (cf. Fig. 2) to the effective gauge field action [6].

As we work in a purely spacelike frame, the usual definition of the phase-space integral is applicable, according to the discussion of Section 3. We only have to perform the standard three-particle phase-space integral. The details of the calculation are relegated to Appendix C.

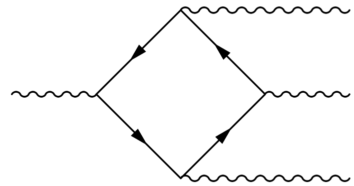

Fig. 2. Feynman diagram contributing to photon triple-splitting (see text).

In the purely spacelike frame ($\zeta^0 = 0$ and $|\boldsymbol{\zeta}| = m$), the decay width is found to be given by

$$\Gamma(\mathbf{q}) = \frac{1}{2\omega(\mathbf{q})}\,\gamma(q_\parallel), \tag{6.4}$$

with the following behavior of the decay parameter for $|q_\parallel| \gg m$:

$$\gamma(q_\parallel) \sim c\,K^2\,m^5|q_\parallel|^5 = c\,(2/45)^2\,\alpha^4\,m^5|q_\parallel|^5/M^8, \tag{6.5}$$

in terms of the model constant K from (6.3) and a numerical constant c depending on the decay channel,

$$c(\oplus \to \ominus\,\ominus\,\ominus) \approx 1.078 \times 10^{-7}, \tag{6.6a}$$

$$c(\ominus \to \ominus\,\ominus\,\ominus) \approx 1.182 \times 10^{-7}, \tag{6.6b}$$

$$c(\oplus \to \oplus\,\ominus\,\ominus) \approx 6.214 \times 10^{-8}. \tag{6.6c}$$

The appearance of a fifth power of the momentum component $|q_\parallel|$ in the asymptotic result (6.5) can be understood as follows. First, the phase-space volume grows linearly with $|q_\parallel|$. Second, the four derivatives in the interaction lead to an amplitude square that is eighth-order in the momenta and the phase-space integral gives a quartic dependence on the effective mass square, which results in a fourth power of $q_\parallel$ (see Section 4.4 for similar results in the scalar model).

It is, however, surprising that there are no suppression effects from the polarizations, since these decays violate angular-momentum conservation. In the decay process $\oplus \to \ominus\ominus\ominus$ with initial momentum $\mathbf{q} \parallel \boldsymbol{\zeta}$, for example, a left-circularly polarized photon turns into three roughly right-circularly polarized photons with the same total energy-momentum.

Cancellations may, however, occur for an interaction with more derivatives. Replacing $\mathcal{S}_{\text{EH}}$ in (6.1) by

$$\mathcal{S}'_{A^4} = K' \int_{\mathbb{R}^4} \mathrm{d}^4x\; F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}\Box F^{\rho\sigma}, \tag{6.7}$$

for example, gives the following additional factors f in the transition amplitude square:

$$f(\oplus \to \ominus\ominus\ominus) = m^2\Big(\omega_{\parallel}(q) + \sum_i \omega_{\parallel,i}\Big)^2 \sim 2m^2 q_{\parallel}^2, \tag{6.8a}$$

$$f(\ominus \to \ominus\ominus\ominus) = m^2\Big(-\omega_{\parallel}(q) + \sum_i \omega_{\parallel,i}\Big)^2 \sim m^4, \tag{6.8b}$$

$$f(\oplus \to \oplus\ominus\ominus) = m^2\Big(\omega_{\parallel}(q) + \omega_{\parallel,1} + \omega_{\parallel,2} - \omega_{\parallel,3}\Big)^2, \tag{6.8c}$$

with the asymptotic behavior for $|q_{\parallel}| \gg m$ shown in two cases. The decay channel $\oplus \to \ominus\ominus\ominus$ then picks up an additional factor $2m^2 q_{\parallel}^2$ compared to the Euler–Heisenberg result, so that $\gamma \sim c'\, K'^2\, m^7\, |q_{\parallel}|^7$. The decay channel $\ominus \to \ominus\ominus\ominus$, on the other hand, has $\gamma \sim c'\, K'^2\, m^9\, |q_{\parallel}|^5$, with two powers of $q_{\parallel}$ suppressed.

Let us return to the decay parameter γ from the original low-energy effective theory (6.1). The validity domain of the effective theory can be expected to be given by $m|q_{\parallel}| \ll 4M^2$, for the particular Lorentz-violating decay process considered. [4] If correct, this would imply that the decay parameter (6.5) grows to a value of order $\alpha^4\, M^2$. But, it is also clear that, as long as $m \ll M$, this would require very large energies on the scale of usual elementary particle physics; see, in particular, Eq. (5.5) of the previous section. Still, there may be entirely different applications of the simple model (6.1), which may turn out to be experimentally accessible, either directly or indirectly. As mentioned before, the intention of the present article is purely theoretical.

With that intention stated, let us comment on the possible high-energy behavior of photon triple-splitting in modified QED (5.1), as our calculation was only valid for momenta less than the electron mass M or perhaps even for momenta less than M^2/m, with an extra factor M/m for Chern–Simons scale m. Recall that, for standard QED, the amplitude of a four-photon interaction to order $\mathrm{O}(\alpha^2)$ is known in principle [6].

For the MCS photon, the triple-splitting phase-space volume grows as $m|q_{\parallel}|$ for $|q_{\parallel}| \to \infty$. But, for a photon-splitting process, it is also known that all scalar products between the photon momenta vanish in standard QED and that the

[4] The range quoted is suggested by two results. First, we have estimated the leading-order correction to (6.5) from Feynman box diagrams (Fig. 2) and find a relative correction of order $m|q_{\parallel}|/M^2$. Second, the pair-creation threshold (5.4) sets a boundary beyond which the Euler–Heisenberg effective action is certainly no longer valid; see the discussion in, e.g., Section 13.2 of Ref. [6]. Note that a similar conclusion on the validity domain of the Euler–Heisenberg effective action in the context of other Lorentz-violating theories appears to have been reached in Ref. [16].

amplitude square is zero [20]. Therefore, the dimensionless MCS decay amplitude square must be of order m^n, for $n \geq 1$, as it is in the low-energy region. However, the precise functional dependence may be rather subtle. There could, for example, be a factor $\Big(m|q_\parallel|/(m|q_\parallel| + M^2)\Big)^4$ in the amplitude square. For the moment, we just make the following conjecture:

$$|A|^2 \,\Big|_{|q_\parallel| \gg 4M^2/m} \overset{?}{\sim} \alpha^4 \,, \tag{6.9}$$

neglecting a possible logarithmic dependence on $m|q_\parallel|/M^2$.

The conjectured behavior (6.9) would imply for the decay parameter:

$$\gamma \,\Big|_{|q_\parallel| \gg 4M^2/m} \overset{?}{\sim} c_\infty \, \alpha^4 \, m \, |q_\parallel| \,. \tag{6.10}$$

Combined with the low-energy result (6.5), this would mean that the effect of Lorentz breaking keeps growing with energy. At ultra-high energies, the decay rate (6.4) would then approach a direction-dependent constant (up to logarithms). A similar behavior has been seen for vacuum Cherenkov radiation; cf. Eq. (5.14).

7 Discussion

In this article, we have considered photon and scalar models with a super-renormalizable term containing a Lorentz-violating background four-vector $\zeta^\mu \equiv m\,\widehat{\zeta}^{\,\mu}$. The modified dispersion relations of these models shift the energy only slightly for high momentum $|\mathbf{q}| \gg m$. In the spacelike Maxwell–Chern–Simons model (2.3), for example, one has

$$\omega(\mathbf{q}) \sim |\mathbf{q}| \pm \,|\cos\theta|\, m/2, \tag{7.1}$$

where θ is the angle between wave vector $\mathbf{q}$ and the background vector $\boldsymbol{\zeta} \equiv m\,\widehat{\boldsymbol{\zeta}}$ in a purely spacelike frame. This dispersion relation is different from other Lorentz-violating dispersion relations considered in the literature (e.g., Refs. [16,22]) where the deviation from Lorentz invariance grows rapidly with momentum.

But even the mild Lorentz violation (7.1) has important effects on the high-energy behavior as the absolute value of the effective mass square (2.17) does increase with momentum. Both the phase-space measure and the derivative terms in the interaction pick up this effective mass square.

For theories with only scalar particles, the Lorentz-violating effects can be understood solely in terms of the modified kinematics. For particles with spin,

the amplitude is also modified by the different tensor structure of the photons; see, in particular, the discussion below Eq. (5.8).

The importance of this kind of Lorentz-symmetry breaking should, however, not be overestimated because observables will still be suppressed by powers of the soft Lorentz-breaking scale m. Moreover, in the low-energy region, the Lorentz violation can be hidden by standard mass terms that dominate the amplitude and kinematics.

Lorentz-noninvariant effects may, on the other hand, become visible if a standard mass term is not allowed (as by gauge invariance in the Maxwell–Chern–Simons model) or if processes are considered that are normally kinematically forbidden (as vacuum Cherenkov radiation and photon triple-splitting in standard quantum electrodynamics). In both cases, the background vector ζ^μ is crucial to obtain a non-zero probability.

Up till now, we have considered one particular modification of quantum electrodynamics (QED), namely the theory (5.1) with an additional photonic Chern–Simons-like term (2.5). Lorentz violation in the electron sector may also lead to a nonzero amplitude for photon-triple splitting [23]. However, without modified photon kinematics, this still does not give a nonzero decay width of the photon because the phase-space volume is zero [24]. On the other hand, if there is a bi-linear Lorentz-violating contribution to the photon action, one typically expects both photon triple-splitting and vacuum Cherenkov radiation to occur.

Apart from the Chern–Simons-like term, there is only one other renormalizable bi-linear Lorentz-violating term in the Standard Model extension [2,25]

$$\mathcal{S}_k = \int_{\mathbb{R}^4} \mathrm{d}^4x \; k^{\mu\nu\rho\sigma} \, F_{\mu\nu} F_{\rho\sigma} \,, \tag{7.2}$$

with a real dimensionless background tensor $k^{\mu\nu\rho\sigma}$. This background tensor $k^{\mu\nu\rho\sigma}$ has certain obvious symmetries (the same ones as the Riemann tensor), is taken to be doubly traceless (so as not to give the standard Maxwell term), and does not contain a totally antisymmetric part (which would produce a total derivative). But, its tensor structure remains complicated and explicit calculations of decay processes are difficult. In the following, we will refer to (7.2) as the k–term.

This CPT–even k–term contains, in fact, two derivatives and yields a dispersion relation with stronger Lorentz violation at large momenta than the CPT–odd Chern–Simons-like term (2.5) with a single derivative. Purely on dimensional grounds, one can write

$$\omega(\mathbf{q})^2 = |\mathbf{q}|^2 \left(1 + \Theta(\widehat{\mathbf{q}})\right), \tag{7.3}$$

with $\widehat{\mathbf{q}} \equiv \mathbf{q}/|\mathbf{q}|$ and a dimensionless function Θ carrying the direction dependence. Typically, this dispersion relation leads to a direction-dependent group velocity, unless Θ is independent of $\widehat{\mathbf{q}}$.

Consistency of the theory demands, most likely, a group velocity of electromagnetic waves not larger than one (here, the maximum attainable velocity of the electrons) and the model will probably only be causal for certain choices of background tensor $k^{\mu\nu\rho\sigma}$. Indeed, one such choice has been presented in Eq. (50) of Ref. [2]. For that particular model, there is one photon mode with lightlike momentum and one mode with spacelike momentum, making both photon triple-splitting and Cherenkov radiation possible in principle.

For QED with the additional photonic term (7.2),

$$\mathcal{S}_{\text{QED}+k\text{–term}} = \mathcal{S}_{\text{QED}} + \mathcal{S}_k \,, \tag{7.4}$$

one expects the triple-splitting decay width to carry a larger power of the momentum than the decay width from the theory (5.1) considered in the rest of this article. The reason is that the k–term (7.2) contains more derivatives than the Chern–Simons-like term (2.5) and has, moreover, no explicit mass scale (the photonic sector violates Lorentz invariance, but, at tree level, still has conformal invariance). In the low-energy region of the photon-triple-splitting process, we conjecture that the leading term of the decay parameter is given by

$$\gamma(\mathbf{q})\,\Big|_{(kqq)\ll M^2} \;\stackrel{?}{\sim}\; \alpha^4\,(kqq)^5\,/M^8 \,, \tag{7.5}$$

where M stands for the electron mass and $(kqq)^5$ is a function with a possibly complicated tensor structure involving $k^{\mu\nu\rho\sigma}$ to the fifth power and q^μ to the tenth. For the asymptotic behavior at large momentum, we conjecture the following momentum dependence (neglecting logarithms):

$$\gamma(\mathbf{q})\,\Big|_{(kqq)\gg M^2} \;\stackrel{?}{\sim}\; \alpha^4\,(kqq)^5/(kqq)^4 \sim\; \alpha^4\,(kqq) \,, \tag{7.6}$$

with the highly symbolic notation (kqq). This conjectured decay parameter (7.6) from the k–term would be qualitatively the same as (6.10) from the Chern–Simons-like term, with possibly interesting implications for the high-energy theory. However, only a complete calculation can tell whether or not (6.10) and (7.6) hold true.

Acknowledgements

It is a pleasure to thank E. Kant and C. Rupp for useful discussions.

A Photon polarization vectors in the MCS model

In this appendix, we present simplified expressions for the polarization vectors of the Maxwell–Chern–Simons (MCS) gauge field in purely spacelike frames. For explicit calculations involving multiple particles, it turns out to be more practical to have them in a momentum-independent basis, unlike the expressions used in Refs. [4,5].

Let $\widehat{\boldsymbol{\xi}}$ be an arbitrary unit vector orthogonal to the background three-vector $\widehat{\boldsymbol{\zeta}}$ and define $\widehat{\boldsymbol{\eta}} \equiv \widehat{\boldsymbol{\zeta}} \times \widehat{\boldsymbol{\xi}}$. Then, use cylinder coordinates with $\widehat{\boldsymbol{\zeta}}$ as the cylinder axis and azimuthal angle ϕ measured away from $\widehat{\boldsymbol{\xi}}$. The momentum three-vector $\mathbf{k}$ can now be written as

$$\mathbf{k} = k_{\parallel}\,\widehat{\boldsymbol{\zeta}} + k_{\perp}(\cos\phi\,\widehat{\boldsymbol{\xi}} + \sin\phi\,\widehat{\boldsymbol{\eta}}) = R_{\phi}\,(k_{\perp}, 0, k_{\parallel})^{\mathrm{T}}, \tag{A.1}$$

with T standing for ‘transpose’ and the rotation matrix

$$R_{\phi} \equiv \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.2}$$

For the gauge field A^{μ}, the polarizations are most transparent in the Lorentz gauge $k_{\mu}\epsilon^{\mu}(k) = 0$:

$$\epsilon^{\mu}_{\pm}(\mathbf{k}) = \frac{1}{\sqrt{(2\,\omega_{\parallel,\pm} \mp m)\,\omega_{\parallel,\pm}}}\;\tilde{R}_{\phi}\left(-k_{\perp}\,,\ -\omega_{\pm}\,,\ \pm\mathrm{i}\,\omega_{\parallel,\pm}\,,\ 0\right)^{\mathrm{T}}, \tag{A.3}$$

which is consistent with the polarization vectors of Ref. [5]. Here, $\tilde{R}_{\phi}$ is the rotation matrix (A.2) extended to four-dimensional spacetime,

$$\tilde{R}_{\phi} \equiv \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & R_{\phi} \end{pmatrix}, \tag{A.4}$$

and we have used the abbreviations (2.8) and (2.9) for the frequencies $\omega_{\pm}$ and $\omega_{\parallel,\pm}$, which are taken to have an implicit argument $\mathbf{k}$.

The explicit polarizations (A.3) are not needed, as long as the calculation produces gauge-invariant expressions of the form

$$\bar{\epsilon}^{\mu}(k)\epsilon^{\nu}(k)\, R_{\mu\nu}\,, \tag{A.5}$$

with $k^{\mu}R_{\mu\nu}=0$ and $k^{\nu}R_{\mu\nu}=0$. One can then simply replace

$$\bar{\epsilon}^{\mu}(k)\,\epsilon^{\nu}(k)\;\mapsto\;\frac{1}{2k^2+\zeta^2}\Big(-k^2\eta^{\mu\nu}-\zeta^{\mu}\zeta^{\nu}+\mathrm{i}\,\epsilon^{\mu\nu\rho\sigma}\zeta_{\rho}k_{\sigma}\Big), \tag{A.6}$$

for the $\oplus$ and $\ominus$ modes separately (provided $m\neq 0$). This is analogous to the replacement

$$\sum_{r}\bar{\epsilon}^{\mu}_{r}(k)\,\epsilon^{\nu}_{r}(k)\;\mapsto\;-\eta^{\mu\nu} \tag{A.7}$$

for standard photons [6]. An expression like (A.6) also appears in the MCS photon propagator, together with terms that vanish on-shell (see, e.g., Ref. [3]).

Equation (A.6) can be proven by establishing the equality in the Lorentz gauge. Remark that, even for large k^2, it is not a good approximation to drop the $\zeta^{\mu}\zeta^{\nu}$ term in (A.6) because $(k^{\mu}\zeta_{\mu})^2$ is still approximately equal to k^4 according to Eq. (2.7).

In principle, (A.6) suffices to calculate every possible gauge-invariant amplitude square, but, for completeness, we also list the electric and magnetic field polarizations as defined in Ref. [4]:

$$\mathbf{f}_{\pm}(\mathbf{k})=N\,R_{\phi}\begin{pmatrix}-\mathrm{i}\,\omega_{\parallel,\pm}\sqrt{\omega_{\parallel,\pm}}\\[4pt] \mp\omega_{\pm}\sqrt{\omega_{\parallel,\pm}}\\[4pt] \mathrm{i}\,\chi\,k_{\perp}\sqrt{\omega_{\parallel,\pm}\mp m}\end{pmatrix},\quad \mathbf{b}_{\pm}(\mathbf{k})=N\,R_{\phi}\begin{pmatrix}\pm\chi\,\omega_{\parallel,\pm}\sqrt{\omega_{\parallel,\pm}\mp m}\\[4pt] -\mathrm{i}\,\chi\,\omega_{\pm}\sqrt{\omega_{\parallel,\pm}\mp m}\\[4pt] \mp k_{\perp}\sqrt{\omega_{\parallel,\pm}}\end{pmatrix}, \tag{A.8}$$

with common normalization factor $N\equiv(m^2+4k_{\parallel}^2)^{-1/4}$ and $\chi\equiv\operatorname{sgn}k_{\parallel}$.

B Kinematics of MCS photon triple-splitting

In this appendix, we show how to solve the energy-momentum conservation condition in photon triple-splitting,

$$\omega(\mathbf{q})\Big|_{\mathbf{q}=\sum_i\mathbf{k}_i}=\sum_{j}\omega(\mathbf{k}_j)\,, \tag{B.1}$$

where $\mathbf{q}$ is the three-momentum of the decaying particle and $\mathbf{k}_i$ $(i=1\ldots 3)$ are those of the decay products. The energies are given by the Maxwell–Chern–

Simons dispersion relation (2.8), i.e., we work in a frame where $\widehat{\zeta}^{\mu}$ is purely spacelike. The considerations of this appendix are independent of the kind of interaction responsible for the splitting process.

As seen in Ref. [4], a direct solution of (B.1) is not possible. The crucial observation, now, is that the purely spacelike theory is still invariant under Lorentz boosts in directions orthogonal to the preferred axis $\widehat{\boldsymbol{\zeta}}$. [The reason is that the orthogonal momentum $\mathbf{k}_{\perp}$ enters (2.8) in the standard way.] There, then, exists a Lorentz transformation which transforms the case $\mathbf{q} \nparallel \widehat{\boldsymbol{\zeta}}$ to one with $\mathbf{q} \parallel \widehat{\boldsymbol{\zeta}}$, except for the special case of a $\ominus$ photon with $\mathbf{q} \perp \widehat{\boldsymbol{\zeta}}$, which will be dealt with later.

For the case $\mathbf{q} \parallel \widehat{\boldsymbol{\zeta}}$, the optimal situation obviously has all momenta aligned because orthogonal components of the $\mathbf{k}_i$ increase the right-hand side of (B.1). Therefore, it suffices to consider the case with all four vectors $\mathbf{q}$ and $\mathbf{k}_i$ parallel to $\widehat{\boldsymbol{\zeta}}$. An elementary calculation shows then that

$$\omega_{+}(\mathbf{q}) < \omega_{+}(\mathbf{k}_1) + \omega_{+}(\mathbf{k}_2) + \omega_{-}(\mathbf{k}_3). \tag{B.2}$$

Hence, the decay channel $\oplus \to \oplus\oplus\ominus$ is kinematically forbidden and the same holds for the channel $\ominus \to \oplus\ominus\ominus$ and those involving more $\oplus$ decay products.

On the other hand, the splitting $\ominus \to \ominus\ominus\ominus$ is allowed, because the following inequality holds for arbitrary momentum $\mathbf{q} \neq \mathbf{0}$:

$$\omega_{-}(\mathbf{q}) > 3\,\omega_{-}(\mathbf{q}/3). \tag{B.3}$$

Now, $\oplus \to \ominus\ominus\ominus$ decay is also possible, as $\omega_{+} > \omega_{-}$ holds generally. By making use of the relation

$$\omega_{+}(\mathbf{q}) = \omega_{-}(\mathbf{q}) + m, \tag{B.4}$$

for $\mathbf{q} \parallel \widehat{\boldsymbol{\zeta}}$, one can furthermore show that the channel $\oplus \to \oplus\ominus\ominus$ is allowed.

There remains one special case to be discussed. If the initial momentum of a $\ominus$ photon is orthogonal to $\widehat{\boldsymbol{\zeta}}$, there is no Lorentz transformation which removes the orthogonal component $\mathbf{q}_{\perp}$. However, for $\mathbf{q} \cdot \widehat{\boldsymbol{\zeta}} = 0$, the dispersion relation of the $\ominus$ photon is that of a usual massless particle while the $\oplus$ mode has mass m. Thus, only the splitting $\ominus \to \ominus\ominus\ominus$ is possible here and the allowed region has zero phase-space volume, which, under quite general assumptions, yields a vanishing decay rate [20]. Similarly, one sees that the decay width for $\oplus \to \oplus\ominus\ominus$ is zero for initial three-momentum orthogonal to $\boldsymbol{\zeta}$.

To summarize, we have established that only three decay channels are allowed for photon triple-splitting: $\oplus \to \oplus\ominus\ominus$, $\ominus \to \ominus\ominus\ominus$ and $\oplus \to \ominus\ominus\ominus$, where the former two are for $\mathbf{q} \cdot \widehat{\boldsymbol{\zeta}} \neq 0$ and the latter one for arbitrary initial

momentum $\mathbf{q}$. Hence, there are no stable $\oplus$ photons in the model (2.3), while the $\ominus$ photons are only stable for a momentum subset of measure zero.

C Phase-space integration for MCS–EH photon triple-splitting

In this appendix, we sketch the calculation of the decay parameter γ for photon triple-splitting in the low-energy effective photon theory (6.1), consisting of the free Maxwell–Chern–Simons model (2.3) with Euler–Heisenberg interaction (6.2).

In the purely spacelike frame, we have to perform a standard phase-space integral,

$$\Gamma(\mathbf{q}) = \frac{1}{\sigma}\frac{1}{2\omega(\mathbf{q})}\int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^3k_i}{(2\pi)^3\,2\omega_i(\mathbf{k}_i)}\right) \times\,(2\pi)^4\delta^3\Big(\mathbf{q}-\sum_j\mathbf{k}_j\Big)\,\delta\Big(\omega(\mathbf{q})-\sum_l\omega_l(\mathbf{k}_l)\Big)\,|A(\mathbf{q},\omega,\mathbf{k}_i,\omega_i)|^2. \tag{C.1}$$

With the Euler–Heisenberg interaction (6.2), the transition amplitude for the splitting of one photon with momentum $\mathbf{q}$ into three with momenta $\mathbf{k}_i$ can be written as [4]

$$A = -\frac{1}{2\pi^2}\,K\,(A_{12}+A_{23}+A_{31}), \tag{C.2}$$

in terms of the coupling constant K from (6.3) and the following expression:

$$A_{ab} = \Big[\Big(\mathbf{f}(\mathbf{k}_a)\cdot\mathbf{f}(\mathbf{k}_b)-\mathbf{b}(\mathbf{k}_a)\cdot\mathbf{b}(\mathbf{k}_b)\Big)\Big(\mathbf{f}(\mathbf{k}_c)\cdot\mathbf{f}^*(\mathbf{q})-\mathbf{b}(\mathbf{k}_c)\cdot\mathbf{b}^*(\mathbf{q})\Big) +\frac{7}{4}\Big(\mathbf{f}(\mathbf{k}_a)\cdot\mathbf{b}(\mathbf{k}_b)+\mathbf{f}(\mathbf{k}_b)\cdot\mathbf{b}(\mathbf{k}_a)\Big)\Big(\mathbf{f}(\mathbf{k}_c)\cdot\mathbf{b}^*(\mathbf{q})+\mathbf{b}(\mathbf{k}_c)\cdot\mathbf{f}^*(\mathbf{q})\Big)\Big], \tag{C.3}$$

which depends on the polarization vectors (A.8) for the considered channel (particle indices a, b, $c\in\{1,2,3\}$ all different).

The evaluation of the phase-space integral for arbitrary initial momentum is simplified by the same argument that helps with solving the kinematics, namely, the result must be invariant with respect to Lorentz boosts in a direction orthogonal to $\widehat{\boldsymbol{\zeta}}$. This suggests splitting the momentum integrations as follows:

$$\frac{\mathrm{d}^3k}{\omega(\mathbf{k})} = \frac{\mathrm{d}^3k}{\sqrt{\omega_\parallel(k_\parallel)^2+k_\perp^2}} = \mathrm{d}k_\parallel\,\frac{\mathrm{d}^2k_\perp}{\sqrt{\mu(k_\parallel)^2+k_\perp^2}}, \tag{C.4}$$

with $\mu(k_\parallel) \equiv \omega_\parallel(k_\parallel)$.[5] The momentum-conservation δ-function is split in the same way. Then, the phase-space integral is factorized into an ordinary integral over the parallel components and a Lorentz-invariant three-particle phase-space integral in $2+1$ dimensions with masses $\mu(k_{\parallel,i})$ and $\mu(q_\parallel)$.

In order to perform the integral over the orthogonal components, the phase-space integrand $|A|^2$ in (C.1) needs to be rewritten in a $(2+1)$–dimensional Lorentz-invariant form, which turns out to be possible. The explicit expressions are, however, quite complicated. We define the $(2+1)$-dimensional Lorentz vectors

$$\lambda_i^\mu \equiv \Big(\sqrt{\mu_i^2 + k_{\perp,i}^2}\,,\ \mathbf{k}_{\perp,i}\Big) \equiv (\omega_i\,, \mathbf{k}_{\perp,i}), \tag{C.5a}$$

$$\lambda_q^\mu \equiv \Big(\sqrt{\mu^2 + q_\perp^2}\,,\ \mathbf{q}_\perp\Big) \equiv (\omega_q\,, \mathbf{q}_\perp). \tag{C.5b}$$

All the factors in the integrand which depend on the orthogonal momenta can then be written as products of contractions of λ_q and λ_i, where each particle "momentum" occurs in second order. Additional factors that only depend on $k_{\parallel,i}$ or $\omega_{\parallel,i}$ are pulled out of the inner integrals.

Next, we perform the Lorentz-invariant phase-space integrals for the $2+1$ dimensions. These can be solved by mass convolutions which, unlike for the $(3+1)$-dimensional case, do not lead to elliptic integrals but to polynomials in the formal masses μ and μ_i.

After this integration, we are effectively left with a double integral over momentum components parallel to $\widehat{\boldsymbol{\zeta}}$ and an integrand that is, except for a factor $1/\omega_\parallel(q_\parallel)$ and the squared normalization factors of the polarization vectors, a polynomial in the momentum components and the parallel energies. Despite this relatively simple structure, we have not been able to perform the integral analytically. However, both by numerical methods and by Laurent expansion (with a low-momentum cutoff as additional approximation), the behavior for large momentum component $|q_\parallel|$ could be extracted. The result has been given in Eq. (6.6) of the main text.

For a $\oplus$ photon at rest ($\mathbf{q} = \mathbf{0}$), the decay parameter of the only open channel $\oplus \to \ominus\,\ominus\,\ominus$ was expressed in Ref. [4] as

$$\gamma_\oplus(0) = 2m\,\frac{K^2\,m^9}{3!\;512\pi^5\;4\pi^4}\;I, \tag{C.6}$$

with a preliminary estimate $I \approx 0.2$ for the dimensionless phase-space integral.

[5] Note that this definition has nothing to do with the effective mass square from Section 2.4.

We have now obtained an improved numerical value for this integral,

$$I \approx 0.02773, \tag{C.7}$$

which is lower by a factor 7 (most likely, the numerical accuracy of Ref. [4] was insufficient to get a reliable estimate). Writing $\gamma_{\oplus}(0) = c_0\, K^2\, m^{10}$, we then have $c_0 \approx 1.514 \times 10^{-10}$ at $q_{\parallel} = 0$. For larger momentum components $|q_{\parallel}|$ and the exclusive decay channel $\oplus \to \ominus\,\ominus\,\ominus$, the decay parameter $\gamma_{\oplus}(q_{\parallel})$ reaches the power-law behavior (6.5) with constant (6.6a), which holds for $m \ll |q_{\parallel}| \ll M^2/m$.

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